Mathematics: analysis and approaches

Higher level

Paper 3

Name

Date: ____________________

1 hour

Instructions to candidates

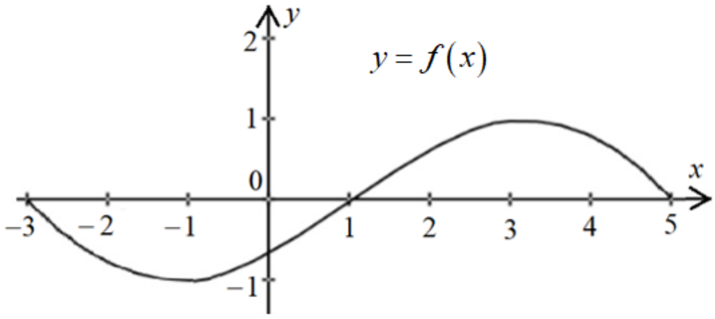
- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

exam: 3 pages

Answer all questions on separate answer paper / booklet. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

The following graph represents a function $y = f(x)$, where $-3 \le x \le 5$. The function has a maximum at $(3,1)$ and a minimum at $(-1,-1)$.



(a) The functions u and v are defined as $u(x) = x-3$ and $v(x) = 2x$ where $x \in \mathbb{R}$.

(i) State the range of the composite function $u \circ f$.

(ii) State the range of the composite function $u \circ v \circ f$.

(iii) Find the largest possible domain of the function $f \circ v \circ u$ [7]

(b) (i) Explain why f does not have an inverse.

(ii) The domain of f is restricted to define a function g so that it has an inverse g^{-1}. State the largest possible domain of g.

(iii) Sketch a graph of $y = g^{-1}(x)$, showing clearly the y-intercept and stating the coordinates of the endpoints. [8]

A function is self-inverse when the function and its inverse are the same function.

Consider the function defined by $h(x) = \dfrac{2x-5}{x+d}$, $x \ne -d$ and $d \in \mathbb{R}$.

(c) (i) Find an expression for the inverse function $h^{-1}(x)$.

(ii) Write down the value of d such that h is a self-inverse function.

For this value of d, there is a function k such that $(h \circ k)(x) = \dfrac{2x}{x+1}$, $x \ne -1$

(ii) Show that $k(x) = \dfrac{x+5}{2}$. [6]

(d) Determine the conditions for which $r(x) = \dfrac{ax+b}{cx+d}$, $x \ne -\dfrac{c}{d}$ is a self-inverse function. [4]

2. [Maximum mark: 30]

Consider the curve $y = x\mathrm{e}^{x}$ and the line $y = kx,\ k \in \mathbb{R}$.

(a) Let $k = 0$. Show that the curve and the line intersect only once. [2]

(b) Let $k = 1$. Show that the line is a tangent to the curve. [2]

(c) (i) Find the values of k such that the curve $y = x\mathrm{e}^{x}$ and the line $y = kx$ intersect at two distinct points.

(ii) Write down the coordinates of the two points of intersection. [5]

(d) Let A be the region enclosed by the curve and the line when $k > 1$.

(i) Write down an integral representing the area of A.

(ii) Find the **exact** area of A when $k = \mathrm{e}^{2}$.

(iii) Show that the area of A when $k = \mathrm{e}^{n},\ n \in \mathbb{R}^{+}$, is equal to $\mathrm{e}^{n}\left(\frac{n^{2}}{2} - n + 1\right) - 1$. [9]

(e) The curve has a horizontal tangent at the point P.

(i) Find the **exact** value of k such that the line $y = kx$ passes through P.

(ii) For the value of k found in part (e)(i), show that the area of the region enclosed by the curve and the line is $1 - \frac{5}{2\mathrm{e}}$. [7]

(e) Let B be the region enclosed by the curve and the line when $0 < k < 1$. Show that the area of B must be less than 1. [5]